AP®

SAMPLE QUESTIONS

AP® Calculus AB and AP® Calculus BC Exam

Originally published in the Fall 2014

AP® Calculus AB and AP® Calculus

BC Curriculum Framework

CollegeBoard

The College Board

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Introduction

These sample exam questions were originally included in the *AP Calculus AB and AP Calculus BC Curriculum Framework,* published in fall 2014. The *AP Calculus AB and AP Calculus BC Course and Exam Description,* which is out now, includes that curriculum framework, along with a new, unique set of exam questions. Because we want teachers to have access to all available questions that support the new exam, we are making those from the fall 2014 curriculum framework available in this supplementary document.

The sample exam questions illustrate the relationship between the curriculum framework and the redesigned *AP Calculus AB Exam and AP Calculus BC Exam*, and they serve as examples of the types of questions that appear on the exam.

Each question is accompanied by a table containing the main learning objective(s), essential knowledge statement(s), and Mathematical Practices for AP Calculus that the question addresses. For multiple-choice questions, an answer key is provided. In addition, each free-response question is accompanied by an explanation of how the relevant Mathematical Practices for AP Calculus can be applied in answering the question. The information accompanying each question is intended to aid in identifying the focus of the question, with the underlying assumption that learning objectives, essential knowledge statements, and MPACs other than those listed may also partially apply. Note that in the cases where multiple learning objectives, essential knowledge statements, or MPACs are provided for a multiple-choice question, the primary one is listed first.

AP Calculus AB Sample Exam Questions

Multiple Choice: Section I, Part A

A calculator may not be used on questions on this part of the exam.

1. $\lim\limits_{x \to \pi} \dfrac{\cos x + \sin(2x) + 1}{x^2 - \pi^2}$ is

(A) $\dfrac{1}{2\pi}$

(B) $\dfrac{1}{\pi}$

(C) 1

(D) nonexistent

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 1.1C: Determine limits of functions.	**EK 1.1C3:** Limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.	**MPAC 1:** Reasoning with definitions and theorems
LO 2.1C: Calculate derivatives.	**EK 2.1C2:** Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric.	**MPAC 3:** Implementing algebraic/computational processes

2. $\lim\limits_{x \to \infty} \dfrac{\sqrt{9x^4 + 1}}{x^2 - 3x + 5}$ is

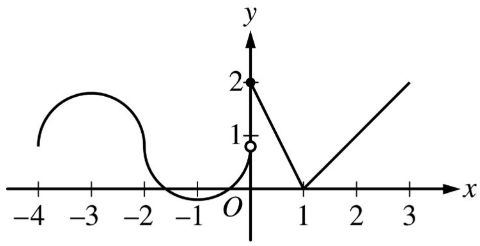
(A) 1

(B) 3

(C) 9

(D) nonexistent

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 1.1C: Determine limits of functions.	**EK 1.1C2:** The limit of a function may be found by using algebraic manipulation, alternate forms of trigonometric functions, or the squeeze theorem.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 2:** Connecting concepts
LO 1.1A(b): Interpret limits expressed symbolically.	**EK 1.1A2:** The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits.	



Graph of f

3. The graph of the piecewise-defined function f is shown in the figure above. The graph has a vertical tangent line at $x = -2$ and horizontal tangent lines at $x = -3$ and $x = -1$. What are all values of x, $-4 < x < 3$, at which f is continuous but not differentiable?

(A) $x = 1$

(B) $x = -2$ and $x = 0$

(C) $x = -2$ and $x = 1$

(D) $x = 0$ and $x = 1$

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.2B: Recognize the connection between differentiability and continuity.	**EK 2.2B1:** A continuous function may fail to be differentiable at a point in its domain.	**MPAC 4:** Connecting multiple representations **MPAC 2:** Connecting concepts
LO 1.2A: Analyze functions for intervals of continuity or points of discontinuity.	**EK 1.2A3:** Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.	

4. An ice sculpture in the form of a sphere melts in such a way that it maintains its spherical shape. The volume of the sphere is decreasing at a constant rate of 2π cubic meters per hour. At what rate, in square meters per hour, is the surface area of the sphere decreasing at the moment when the radius is 5 meters? (Note: For a sphere of radius r, the surface area is $4\pi r^2$ and the volume is $\frac{4}{3}\pi r^3$.)

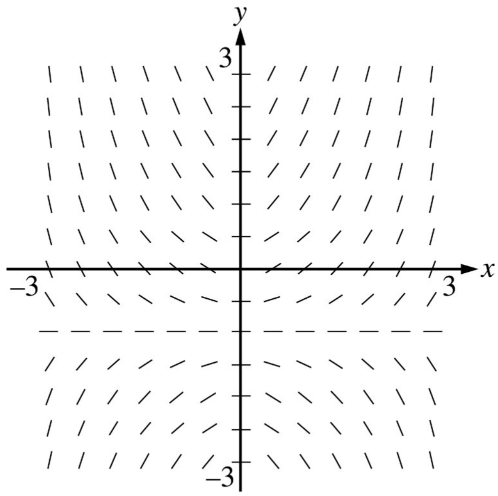
(A) $\frac{4\pi}{5}$

(B) 40π

(C) $80\pi^2$

(D) 100π

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, *(BC)* and planar motion.	**EK 2.3C2:** The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes
LO 2.1C: Calculate derivatives.	**EK 2.1C5:** The chain rule is the basis for implicit differentiation.	



5. Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = xy + x$

(B) $\frac{dy}{dx} = xy + y$

(C) $\frac{dy}{dx} = y + 1$

(D) $\frac{dy}{dx} = (x + 1)^2$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.3F: Estimate solutions to differential equations.	**EK 2.3F1:** Slope fields provide visual clues to the behavior of solutions to first order differential equations.	**MPAC 4:** Connecting multiple representations **MPAC 2:** Connecting concepts

$$f(x) = \begin{cases} 2x - 2 & \text{for } x < 3 \\ 2x - 4 & \text{for } x \geq 3 \end{cases}$$

6. Let f be the piecewise-linear function defined above. Which of the following statements are true?

I. $\displaystyle\lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h} = 2$

II. $\displaystyle\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} = 2$

III. $f'(3) = 2$

(A) None

(B) II only

(C) I and II only

(D) I, II, and III

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient.	**EK 2.1A2:** The instantaneous rate of change of a function at a point can be expressed by $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ or $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided that the limit exists. These are common forms of the definition of the derivative and are denoted $f'(a)$.	**MPAC 2:** Connecting concepts **MPAC 5:** Building notational fluency
LO 1.1A(b): Interpret limits expressed symbolically.	**EK 1.1A2:** The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits.	

7. If $f(x) = \int_1^{x^3} \frac{1}{1 + \ln t}\, dt$ for $x \ge 1$, then $f'(2) =$

(A) $\frac{1}{1 + \ln 2}$

(B) $\frac{12}{1 + \ln 2}$

(C) $\frac{1}{1 + \ln 8}$

(D) $\frac{12}{1 + \ln 8}$

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.3A: Analyze functions defined by an integral.	**EK 3.3A2:** If f is a continuous function on the interval $[a, b]$, then $\frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x)$, where x is between a and b.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 3:** Implementing algebraic/computational processes
LO 2.1C: Calculate derivatives.	**EK 2.1C4:** The chain rule provides a way to differentiate composite functions.	

8. Which of the following limits is equal to $\int_3^5 x^4 \, dx$?

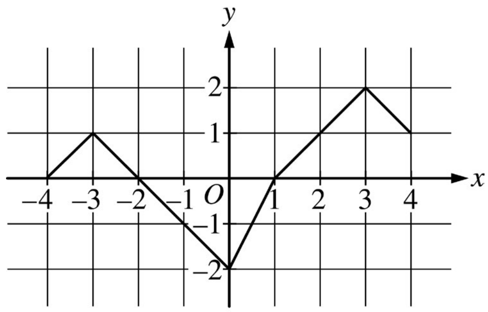
(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{2}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{2}{n}$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.2A(a): Interpret the definite integral as the limit of a Riemann sum.	**EK 3.2A3:** The information in a definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 5:** Building notational fluency



Graph of f

9. The function f is continuous for $-4 \le x \le 4$. The graph of f shown above consists of five line segments. What is the average value of f on the interval $-4 \le x \le 4$?

(A) $\frac{1}{8}$

(B) $\frac{3}{16}$

(C) $\frac{15}{16}$

(D) $\frac{3}{2}$

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.4B: Apply definite integrals to problems involving the average value of a function.	**EK 3.4B1:** The average value of a function f over an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 4:** Connecting multiple representations
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**EK 3.2C1:** In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.	

t	0	2
$f(t)$	4	12

10. Let $y = f(t)$ be a solution to the differential equation $\frac{dy}{dt} = ky,$ where k is a constant. Values of f for selected values of t are given in the table above. Which of the following is an expression for $f(t)$?

(A) $4e^{\frac{t}{2}\ln 3}$

(B) $e^{\frac{t}{2}\ln 9} + 3$

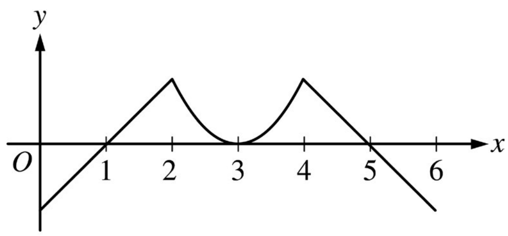
(C) $2t^2 + 4$

(D) $4t + 4$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.5B: Interpret, create and solve differential equations from problems in context.	**EK 3.5B1:** The model for exponential growth and decay that arises from the statement "The rate of change of a quantity is proportional to the size of the quantity" is $\frac{dy}{dt} = ky.$	**MPAC 3:** Implementing algebraic/computational processes **MPAC 4:** Connecting multiple representations

Multiple Choice: Section I, Part B

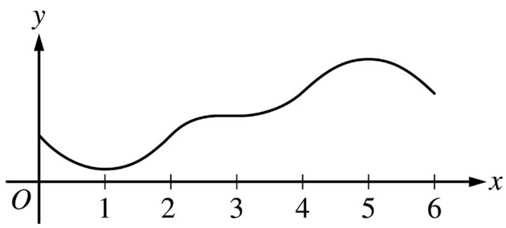
A graphing calculator is required for some questions on this part of the exam.



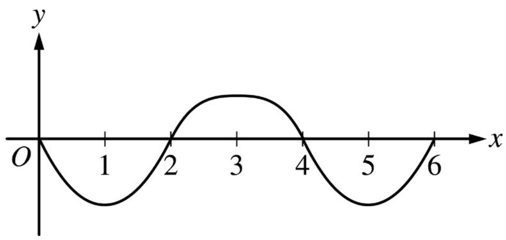
Graph of f'

11. The graph of f', the derivative of the function f, is shown above. Which of the following could be the graph of f?

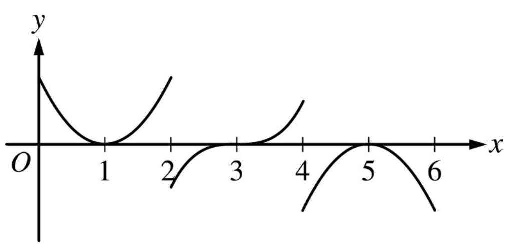
(A)



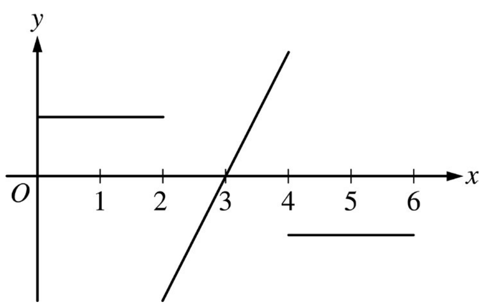
(B)



(C)



(D)



Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A3:** Key features of the graphs of f, f', and f'' are related to one another.	**MPAC 4:** Connecting multiple representations **MPAC 2:** Connecting concepts
LO 2.2B: Recognize the connection between differentiability and continuity.	**EK 2.2B2:** If a function is differentiable at a point, then it is continuous at that point.	

12. The derivative of a function f is given by $f'(x) = e^{\sin x} - \cos x - 1$ for $0 < x < 9$. On what intervals is f decreasing?

(A) $0 < x < 0.633$ and $4.115 < x < 6.916$

(B) $0 < x < 1.947$ and $5.744 < x < 8.230$

(C) $0.633 < x < 4.115$ and $6.916 < x < 9$

(D) $1.947 < x < 5.744$ and $8.230 < x < 9$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	**MPAC 4:** Connecting multiple representations **MPAC 2:** Connecting concepts

13. The temperature of a room, in degrees Fahrenheit, is modeled by H, a differentiable function of the number of minutes after the thermostat is adjusted. Of the following, which is the best interpretation of $H'(5) = 2$?

(A) The temperature of the room is 2 degrees Fahrenheit, 5 minutes after the thermostat is adjusted.

(B) The temperature of the room increases by 2 degrees Fahrenheit during the first 5 minutes after the thermostat is adjusted.

(C) The temperature of the room is increasing at a constant rate of $\frac{2}{5}$ degree Fahrenheit per minute.

(D) The temperature of the room is increasing at a rate of 2 degrees Fahrenheit per minute, 5 minutes after the thermostat is adjusted.

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.3A: Interpret the meaning of a derivative within a problem.	**EK 2.3A1:** The unit for $f'(x)$ is the unit for *f* divided by the unit for *x*.	**MPAC 2:** Connecting concepts **MPAC 5:** Building notational fluency
LO 2.3D: Solve problems involving rates of change in applied contexts.	**EK 2.3D1:** The derivative can be used to express information about rates of change in applied contexts.	

14. A function f is continuous on the closed interval $[2, 5]$ with $f(2) = 17$ and $f(5) = 17$. Which of the following additional conditions guarantees that there is a number c in the open interval $(2, 5)$ such that $f'(c) = 0$?

(A) No additional conditions are necessary.

(B) f has a relative extremum on the open interval $(2, 5)$.

(C) f is differentiable on the open interval $(2, 5)$.

(D) $\int_2^5 f(x)\,dx$ exists.

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.4A: Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**EK 2.4A1:** If a function f is continuous over the interval $[a, b]$ and differentiable over the interval (a, b), the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 5:** Building notational fluency

15. A rain barrel collects water off the roof of a house during three hours of heavy rainfall. The height of the water in the barrel increases at the rate of $r(t) = 4t^3e^{-1.5t}$ feet per hour, where t is the time in hours since the rain began. At time $t = 1$ hour, the height of the water is 0.75 foot. What is the height of the water in the barrel at time $t = 2$ hours?

(A) 1.361 ft

(B) 1.500 ft

(C) 1.672 ft

(D) 2.111 ft

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.4E: Use the definite integral to solve problems in various contexts.	**EK 3.4E1:** The definite integral can be used to express information about accumulation and net change in many applied contexts.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes
LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B2:** If f is continuous on the interval $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.	

16. A race car is traveling on a straight track at a velocity of 80 meters per second when the brakes are applied at time $t = 0$ seconds. From time $t = 0$ to the moment the race car stops, the acceleration of the race car is given by $a(t) = -6t^2 - t$ meters per second per second. During this time period, how far does the race car travel?

(A) 188.229 m

(B) 198.766 m

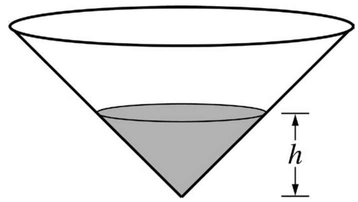
(C) 260.042 m

(D) 267.089 m

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.4C: Apply definite integrals to problems involving motion.	**EK 3.4C1:** For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes
LO 3.1A: Recognize antiderivatives of basic functions.	**EK 3.1A2:** Differentiation rules provide the foundation for finding antiderivatives.	

Free Response: Section II, Part A

A graphing calculator is required for problems on this part of the exam.



1. The height of the water in a conical storage tank, shown above, is modeled by a differentiable function h, where $h(t)$ is measured in meters and t is measured in hours. At time $t = 0$, the height of the water in the tank is 25 meters. The height is changing at the rate $h'(t) = 2 - \dfrac{24e^{-0.025t}}{t+4}$ meters per hour for $0 \le t \le 24$.

   (a) When the height of the water in the tank is h meters, the volume of water is $V = \frac{1}{3}\pi h^3$. At what rate is the volume of water changing at time $t = 0$? Indicate units of measure.

   (b) What is the minimum height of the water during the time period $0 \le t \le 24$? Justify your answer.

   (c) The line tangent to the graph of h at $t = 16$ is used to approximate the height of the water in the tank. Using the tangent line approximation, at what time t does the height of the water return to 25 meters?

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.1C: Calculate derivatives.	**EK 2.1C5:** The chain rule is the basis for implicit differentiation.	**MPAC 1:** Reasoning with definitions and theorems
LO 2.3A: Interpret the meaning of a derivative within a problem.	**EK 2.3A1:** The unit for $f'(x)$ is the unit for *f* divided by the unit for *x*.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes
LO 2.3B: Solve problems involving the slope of a tangent line.	**EK 2.3B2:** The tangent line is the graph of a locally linear approximation of the function near the point of tangency.	**MPAC 5:** Building notational fluency
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, *(BC)* and planar motion.	**EK 2.3C2:** The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.	**MPAC 6:** Communicating

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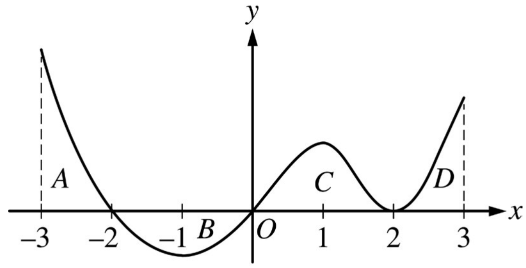
Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, *(BC)* and planar motion.	**EK 2.3C3:** The derivative can be used to solve optimization problems, that is, finding a maximum or minimum value of a function over a given interval.	
LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B2:** If *f* is continuous on the interval $[a, b]$ and F is an antiderivative of *f*, then $\int_a^b f(x)\,dx = F(b) - F(a)$.	
LO 3.4E: Use the definite integral to solve problems in various contexts.	**EK 3.4E1:** The definite integral can be used to express information about accumulation and net change in many applied contexts.	

To answer Question 1 successfully, students must apply the Mathematical Practices for AP Calculus as described below:

- Engage in **reasoning with theorems** (MPAC 1) in order to find the derivative of volume with respect to time as well as in using the Fundamental Theorem of Calculus to find $h(t)$ for particular values of t.
- **Connect the concept** (MPAC 2) of derivative to both the concept of optimization and the concept of slope of a tangent line.
- Use proper **notational fluency** (MPAC 5) to **communicate** (MPAC 6) the process of finding the values for $h(24)$ and $h(6.261)$ and to interpret the meaning of $h'(t)$.
- Use **algebraic manipulation** (MPAC 3) to substitute $\frac{dh}{dt}$ into the expression for $\frac{dV}{dt}$ and find the equation of a tangent line.

Free Response: Section II, Part B

No calculator is allowed for problems on this part of the exam.



Graph of f

2. The graph of a differentiable function f is shown above for $-3 \le x \le 3$. The graph of f has horizontal tangent lines at $x = -1$, $x = 1$, and $x = 2$. The areas of regions A, B, C, and D are 5, 4, 5, and 3, respectively. Let g be the antiderivative of f such that $g(3) = 7$.

   (a) Find all values of x on the open interval $-3 < x < 3$ for which the function g has a relative maximum. Justify your answer.

   (b) On what open intervals contained in $-3 < x < 3$ is the graph of g concave up? Give a reason for your answer.

   (c) Find the value of $\lim_{x \to 0} \dfrac{g(x)+1}{2x}$, or state that it does not exist. Show the work that leads to your answer.

   (d) Let h be the function defined by $h(x) = 3f(2x+1) + 4$. Find the value of $\int_{-2}^{1} h(x)\, dx$.

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 1.1C: Determine limits of functions.	**EK 1.1C3:** Limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 4:** Connecting multiple representations **MPAC 5:** Building notational fluency **MPAC 6:** Communicating
LO 1.2A: Analyze functions for intervals of continuity or points of discontinuity.	**EK 1.2A1:** A function f is continuous at $x = c$ provided that $f(c)$ exists, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$.	
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	
LO 2.2B: Recognize the connection between differentiability and continuity.	**EK 2.2B2:** If a function is differentiable at a point, then it is continuous at that point.	

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**EK 3.2C1:** In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.	
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals.	**EK 3.2C2:** Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.	
LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B5:** Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, *(BC)* integration by parts, and nonrepeating linear partial fractions.	

To answer Question 2 successfully, students must apply the Mathematical Practices for AP Calculus as described below:

- **Reason with definitions and theorems** (MPAC 1) by applying the Fundamental Theorem of Calculus and the concept of area to find the integral over specific intervals.
- Confirm that the hypotheses have been satisfied when applying L'Hospital's rule to find a limit. Correctly using L'Hospital's rule involves **manipulating algebraic** (MPAC 3) quantities.
- **Connect the concepts** (MPAC 2) of a function and its derivative to identify a maximum value and to determine concavity, and **connect the concepts** (MPAC 2) of continuity and limit to find $g(0)$.
- **Connect the graphical representation** (MPAC 4) of a function to the words describing certain attributes of the function and to a symbolic description involving the function.
- Extract information from the graph of $f(x)$ to **compute** (MPAC 3) definite integrals for f and h over specified intervals.
- Build **notational fluency** (MPAC 5) when using integration by substitution to find the integral of $h(x) = 3f(2x+1)+4$ over an interval, including adjusting the endpoints of the interval.
- Clearly **communicate** (MPAC 6) the justification for why a critical point is a relative maximum and indicate the direction of concavity.

Answers and Rubrics (AB)

Answers to Multiple-Choice Questions

1.	B
2.	B
3.	C
4.	A
5.	A
6.	B
7.	D
8.	D
9.	B
10.	A
11.	A
12.	A
13.	D
14.	C
15.	D
16.	B

Rubrics for Free-Response Questions

Question 1

Solutions | **Point Allocation**

(a) $\dfrac{dV}{dt} = \dfrac{1}{3}\pi 3h^2 \dfrac{dh}{dt} = \pi h^2 \dfrac{dh}{dt}$

At $t = 0$,

$\dfrac{dV}{dt} = \pi(25)^2(-4) = -2500\pi = -7853.982$ (or -7853.981) cubic meters per hour.

$2 : \begin{cases} 1 : \dfrac{dV}{dt} \\ 1 : \text{answer with units} \end{cases}$

(b) The absolute minimum must be at a critical point or an endpoint.

$h'(t) = 0$ when $t = 6.261$.

$h(t) = 25 + \int_0^t h'(x)\,dx$

t	$h(t)$
0	25
6.261	16.33873
24	34.56246

The minimum height is 16.339 (or 16.338) meters.

$4 : \begin{cases} 1 : \text{considers } h'(t) = 0 \\ 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{absolute minimum value} \\ 1 : \text{justification} \end{cases}$

(c) $h(16) = 25 + \int_0^{16} h'(t)\,dt = 23.49607$

$h'(16) = 1.19562$

An equation for the tangent line is

$y = 1.196(t - 16) + 23.496$.

$y = 25$ when $t = 17.258$ (or 17.257).

$3 : \begin{cases} 1 : h(16) \\ 1 : \text{tangent line equation} \\ 1 : \text{answer} \end{cases}$

Question 2

Solutions	Point Allocation
(a) g has a relative maximum at $x=-2$ since $g'=f$ changes sign from positive to negative at $x=-2$.	1 : answer with justification
(b) The graph of g is concave up for $-1<x<1$ and $2<x<3$ because $g'=f$ is increasing on those intervals.	$2:\begin{cases}1: \text{answer}\\ 1: \text{reason}\end{cases}$
(c) Because g is continuous at $x=0$, $\lim\limits_{x\to 0} g(x)=g(0)$. $g(3)=g(0)+\int_0^3 f(x)\,dx$ $g(0)=g(3)-\int_0^3 f(x)\,dx=7-(5+3)=-1$ $\lim\limits_{x\to 0} g(x)+1=0$ and $\lim\limits_{x\to 0} 2x=0$. Using L'Hospital's Rule, $\lim\limits_{x\to 0}\frac{g(x)+1}{2x}=\lim\limits_{x\to 0}\frac{g'(x)}{2}=\lim\limits_{x\to 0}\frac{f(x)}{2}=\frac{f(0)}{2}=0$	$3:\begin{cases}1: g(0)\\ 1: \text{L'Hospital's Rule}\\ 1: \text{answer}\end{cases}$
(d) $\int_{-2}^{1} h(x)\,dx=\int_{-2}^{1}(3f(2x+1)+4)\,dx=3\int_{-2}^{1} f(2x+1)\,dx+\int_{-2}^{1} 4\,dx$ Let $u=2x+1$. Then $du=2dx$ and $3\int_{-2}^{1} f(2x+1)\,dx+\int_{-2}^{1} 4\,dx=\frac{3}{2}\int_{-3}^{3} f(u)\,du+12$ $=\frac{3}{2}(5-4+5+3)+12=25.5$	$3:\begin{cases}2: \text{Fundamental Theorem of Calculus}\\ 1: \text{answer}\end{cases}$

AP Calculus BC Sample Exam Questions

Multiple Choice: Section I, Part A

A calculator may not be used on questions on this part of the exam.

1. The position of a particle moving in the xy-plane is given by the parametric equations $x(t) = \dfrac{6t}{t+1}$ and $y(t) = \dfrac{-8}{t^2+4}$. What is the slope of the line tangent to the path of the particle at the point where $t = 2$?

(A) $\dfrac{1}{2}$

(B) $\dfrac{2}{3}$

(C) $\dfrac{3}{4}$

(D) $\dfrac{4}{3}$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.1C: Calculate derivatives.	**EK 2.1C7:** *(BC)* Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions, parametric functions, and functions in polar coordinates.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 2:** Connecting concepts

2. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = 1 + 2y$ with the initial condition $f(0) = 1$. What is the approximation for $f(1)$ if Euler's method is used, starting at $x = 0$ with a step size of 0.5?

(A) 2.5

(B) 3.5

(C) 4.0

(D) 5.5

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.3F: Estimate solutions to differential equations.	**EK 2.3F2:** *(BC)* For differential equations, Euler's method provides a procedure for approximating a solution or a point on a solution curve.	**MPAC 3:** Implementing algebraic/computational processes **MPAC 2:** Connecting concepts

3. For what value of k, if any, is $\int_0^\infty kxe^{-2x}\,dx = 1$?

(A) $\frac{1}{4}$

(B) 1

(C) 4

(D) There is no such value of k.

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.2D: *(BC)* Evaluate an improper integral or show that an improper integral diverges.	**EK 3.2D2:** *(BC)* Improper integrals can be determined using limits of definite integrals.	**MPAC 3:** Implementing algebraic/computational processes
LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B5:** Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, *(BC)* integration by parts, and nonrepeating linear partial fractions.	**MPAC 1:** Reasoning with definitions and theorems

4. The Taylor series for a function f about $x = 0$ converges to f for $-1 \le x \le 1$. The nth-degree Taylor polynomial for f about $x = 0$ is given by $P_n(x) = \sum_{k=1}^{n} (-1)^k \frac{x^k}{k^2 + k + 1}$. Of the following, which is the smallest number M for which the alternating series error bound guarantees that $|f(1) - P_4(1)| \le M$?

(A) $\frac{1}{5!} \cdot \frac{1}{31}$

(B) $\frac{1}{4!} \cdot \frac{1}{21}$

(C) $\frac{1}{31}$

(D) $\frac{1}{21}$

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 4.1B: Determine or estimate the sum of a series.	**EK 4.1B2:** If an alternating series converges by the alternating series test, then the alternating series error bound can be used to estimate how close a partial sum is to the value of the infinite series.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 5:** Building notational fluency
LO 4.2B: Write a power series representing a given function.	**EK 4.2B4:** A Taylor polynomial for $f(x)$ is a partial sum of the Taylor series for $f(x)$.	

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	–2	1	4
1	2	–3	3	–2
2	–1	1	4	5

5. Selected values of a function f and its first three derivatives are indicated in the table above. What is the third-degree Taylor polynomial for f about $x=1$?

(A) $2-3x+\frac{3}{2}x^2-\frac{1}{3}x^3$

(B) $2-3(x-1)+\frac{3}{2}(x-1)^2-\frac{1}{3}(x-1)^3$

(C) $2-3(x-1)+\frac{3}{2}(x-1)^2-\frac{2}{3}(x-1)^3$

(D) $2-3(x-1)+3(x-1)^2-2(x-1)^3$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 4.2A: Construct and use Taylor polynomials.	**EK 4.2A1:** The coefficient of the nth-degree term in a Taylor polynomial centered at $x=a$ for the function f is $\frac{f^{(n)}(a)}{n!}$.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 4:** Connecting multiple representations

6. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ is true?

(A) The series converges absolutely.

(B) The series converges conditionally.

(C) The series converges but neither conditionally nor absolutely.

(D) The series diverges.

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A4:** A series may be absolutely convergent, conditionally convergent, or divergent.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 5:** Building notational fluency

Multiple Choice: Section I, Part B

A graphing calculator is required for some questions on this part of the exam.

7. At time $t \geq 0,$ a particle moving in the *xy*-plane has velocity vector given by $v(t) = \left\langle 4e^{-t}, \sin\left(1+\sqrt{t}\right)\right\rangle$. What is the total distance the particle travels between $t = 1$ and $t = 3$?

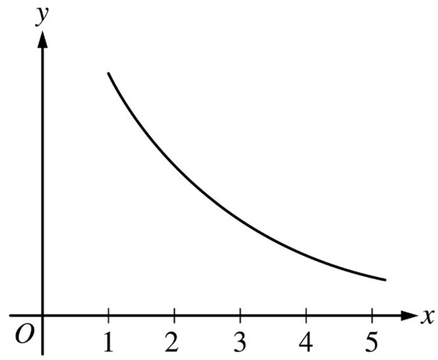
(A) 1.861

(B) 1.983

(C) 2.236

(D) 4.851

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 3.4C: Apply definite integrals to problems involving motion.	**EK 3.4C2:** *(BC)* The definite integral can be used to determine displacement, distance, and position of a particle moving along a curve given by parametric or vector-valued functions.	**MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes



Graph of g

8. For $x \geq 1,$ the continuous function g is decreasing and positive. A portion of the graph of g is shown above. For $n \geq 1$, the nth term of the series $\sum_{n=1}^{\infty} a_n$ is defined by $a_n = g(n)$. If $\int_1^{\infty} g(x)\, dx$ converges to 8, which of the following could be true?

(A) $\sum_{n=1}^{\infty} a_n = 6$

(B) $\sum_{n=1}^{\infty} a_n = 8$

(C) $\sum_{n=1}^{\infty} a_n = 10$

(D) $\sum_{n=1}^{\infty} a_n$ diverges

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A6:** In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the *n*th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 4:** Connecting multiple representations

Free Response: Section II, Part A

A graphing calculator is required for problems on this part of the exam.

1. At time $t \geq 0$, the position of a particle moving along a curve in the xy-plane is $(x(t), y(t))$, where $\frac{dx}{dt} = t - 5\cos t$ and $\frac{dy}{dt} = 6\cos(1 + \sin t)$. At time $t = 3$, the particle is at position $(-1, 2)$.

   (a) Write an equation for the line tangent to the path of the particle at time $t = 3$.

   (b) Find the time t when the line tangent to the path of the particle is vertical. Is the direction of motion of the particle up or down at that moment? Give a reason for your answer.

   (c) Find the y-coordinate of the particle's position at time $t = 0$.

   (d) Find the total distance traveled by the particle for $0 \leq t \leq 3$.

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 2.1C: Calculate derivatives.	**EK 2.1C7:** *(BC)* Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions, parametric functions, and functions in polar coordinates.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 5:** Building notational fluency **MPAC 6:** Communicating
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	
LO 2.3B: Solve problems involving the slope of a tangent line.	**EK 2.3B1:** The derivative at a point is the slope of the line tangent to a graph at that point on the graph.	
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, *(BC)* and planar motion.	**EK 2.3C4:** *(BC)* Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along curves given by parametric or vector-valued functions.	
LO 3.3B(b): Evaluate definite integrals.	**EK 3.3B2:** If f is continuous on the interval $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.	
LO 3.4C: Apply definite integrals to problems involving motion.	**EK 3.4C2:** *(BC)* The definite integral can be used to determine displacement, distance, and position of a particle moving along a curve given by parametric or vector-valued functions.	

To answer Question 1 successfully, students must apply the Mathematical Practices for AP Calculus as described below:

- Engage in **reasoning with definitions and theorems** (MPAC 1) when finding the total distance traveled.
- **Connect the concepts** (MPAC 2) of derivative and position of a particle as well as the concepts of vertical tangent lines and motion.
- Use **algebraic manipulation** (MPAC 3) to find $\frac{dy}{dx}$ and the equation of a tangent line.
- Build **notational fluency** (MPAC 5) by expressing $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and in **communicating** (MPAC 6) the process that leads to finding the y-coordinate and the total distance.
- **Communicate** (MPAC 6) using accurate and precise language and **notation** (MPAC 5) in reporting information provided by technology and in explaining what the sign of $\left.\frac{dy}{dt}\right|_{t=3}$ implies about the vertical direction of motion of the particle.

Free Response: Section II, Part B

No calculator is allowed for problems on this part of the exam.

2. The function f has derivatives of all orders at $x = 0$, and the Maclaurin series for f is $\sum_{n=2}^{\infty} \frac{\ln n}{3^n n^3} x^n$.

   (a) Find $f'(0)$ and $f^{(4)}(0)$.

   (b) Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

   (c) Using the ratio test, determine the interval of convergence of the Maclaurin series for f. Justify your answer.

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
LO 1.1A(b): Interpret limits expressed symbolically.	**EK 1.1A2:** The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits.	**MPAC 1:** Reasoning with definitions and theorems **MPAC 2:** Connecting concepts **MPAC 3:** Implementing algebraic/computational processes **MPAC 5:** Building notational fluency **MPAC 6:** Communicating
LO 2.2A: Use derivatives to analyze properties of a function.	**EK 2.2A1:** First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A3:** Common series of numbers include geometric series, the harmonic series, and *p*-series.	
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A5:** If a series converges absolutely, then it converges.	
LO 4.1A: Determine whether a series converges or diverges.	**EK 4.1A6:** In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the *n*th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.	
LO 4.2A: Construct and use Taylor polynomials.	**EK 4.2A1:** The coefficient of the nth-degree term in a Taylor polynomial centered at $x = a$ for the function f is $\frac{f^{(n)}(a)}{n!}$.	
LO 4.2C: Determine the radius and interval of convergence of a power series.	**EK 4.2C2:** The ratio test can be used to determine the radius of convergence of a power series.	

To answer Question 2 successfully, students must apply the Mathematical Practices for AP Calculus as described below:

- Engage in **reasoning with the definition** (MPAC 1) of the coefficients of the Maclaurin series to find the coefficients for $f'(0)$ and $f^{(4)}(0)$ and in applying the ratio test and comparison test to determine convergence.
- **Connect the concepts** (MPAC 2) of the first and second derivative to find a relative minimum and the concepts of convergence and absolute convergence when finding the interval of convergence.
- Use **algebraic manipulation** (MPAC 3) including working with logarithms and functions to find specific coefficients in the Maclaurin series, the limit in the ratio test, and the interval of convergence.

- Display facility with **notation** (MPAC 5) in **communicating** (MPAC 6) the justification for why f has a relative minimum and what constitutes the interval of convergence.

Answers and Rubrics (BC)

Answers to Multiple-Choice Questions

1.	**C**
2.	**D**
3.	**C**
4.	**C**
5.	**B**
6.	**B**
7.	**A**
8.	**C**

Rubrics for Free-Response Questions

Question 1

Solutions	Point Allocation
(a) $\left.\frac{dy}{dx}\right\|_{t=3} = \left.\frac{dy/dt}{dx/dt}\right\|_{t=3} = \left.\frac{6\cos(1+\sin t)}{t-5\cos t}\right\|_{t=3} = 0.314$ An equation for the tangent line is $y = 2 + 0.314(x+1)$.	$2: \begin{cases} 1: \text{considers } \frac{dy}{dx} \text{ at } t=3 \\ 1: \text{tangent line equation} \end{cases}$
(b) The tangent line is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. $\frac{dx}{dt} = 0$ when $t = 1.30644$. Because $y'(1.30644) = -2.305884 < 0$, the y-coordinate is decreasing and so the particle is moving down at that moment.	$3: \begin{cases} 1: \text{considers } \frac{dx}{dt} = 0 \\ 1: t = 1.30644 \\ 1: \text{conclusion with reason} \end{cases}$
(c) $y(3) = y(0) + \int_0^3 y'(t)\, dt$ $y(0) = y(3) - \int_0^3 y'(t)\, dt = y(3) + 1.63359 = 3.634$ (or 3.633)	$2: \begin{cases} 1: \text{Fundamental Theorem of Calculus} \\ 1: \text{answer} \end{cases}$
(d) Distance $= \int_0^3 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 13.453$	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

Question 2

Solutions	Point Allocation
(a) $\dfrac{f'(0)}{1!} = a_1 = 0 \Rightarrow f'(0) = 0$ $\dfrac{f^{(4)}(0)}{4!} = a_4 = \dfrac{\ln 4}{3^4 4^3} \Rightarrow f^{(4)}(0) = \dfrac{\ln 4}{3^4 4^3} \cdot 4! = \dfrac{\ln 4}{216}$	$2 : \begin{cases} 1 : f'(0) \\ 1 : f^{(4)}(0) \end{cases}$
(b) $f'(0) = 0$ $\dfrac{f''(0)}{2!} = a_2 = \dfrac{\ln 2}{3^2 2^3} \Rightarrow f''(0) = \dfrac{\ln 2}{3^2 2^3} \cdot 2! = \dfrac{\ln 2}{36} > 0$ By the Second Derivative Test, f has a relative minimum at $x = 0$.	$2 : \begin{cases} 1 : \text{considers } f''(0) \\ 1 : \text{answer with justification} \end{cases}$
(c) Using the ratio test, $\lim\limits_{n\to\infty} \left\| \dfrac{\dfrac{\ln(n+1)}{3^{n+1}(n+1)^3} x^{n+1}}{\dfrac{\ln n}{3^n n^3} x^n} \right\| = \lim\limits_{n\to\infty} \left\| \dfrac{\ln(n+1)}{\ln n} \cdot \left(\dfrac{n}{n+1}\right)^3 \cdot \dfrac{x}{3} \right\| = \left\| \dfrac{x}{3} \right\| < 1$ $\|x\| < 3$, therefore the radius of convergence is $R = 3$, and the series converges on the interval $-3 < x < 3$. When $x = 3$, the series is $\sum\limits_{n=2}^{\infty} \dfrac{\ln n}{n^3}$. Because $0 < \dfrac{\ln n}{n^3} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all $n \ge 2$ and the p-series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges, the series $\sum\limits_{n=2}^{\infty} \dfrac{\ln n}{n^3}$ converges by the comparison test.	$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

When $x = -3$, the series is $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n^3}$. This series is absolutely convergent because $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ converges. The interval of convergence is $-3 \le x \le 3$.	